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Exact solutions to the foam drainage equation by using the new generalized (G'/G)-expansion method

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A B S T R A C T

The new generalized (G'/G)-expansion method is an interesting approach to find new and more general exact solutions to the nonlinear evolution equations (NLEEs) in mathematical physics and engineering. In this paper, the method is applied to construct exact solutions involving parameters for the foam drainage equation. When these parameters are taken to be special values, the solitary wave solutions, the periodic wave and the rational function solutions are derived from exact solutions. These solutions might be imperative and significant for the explanation of some practical physical phenomena. It is shown that the method is an easy and advanced mathematical tool for solving NLEEs.

1. Introduction

Since the world around us is inherently nonlinear and NLEEs are widely used to describe compound corporeal phenomena in various fields of sciences, especially in mathematical physics and engineering such as mathematical biology, magneto fluid dynamics, water surface gravity waves, electromagnetic radiation reactions, ion acoustic waves in plasma, fluid mechanics, chemical kinematics, geochemistry, bio-genetics, acoustics, chemistry, biology, protein chemistry etc, many powerful methods to seek exact solutions of NLEEs have been proposed. Among these are the inverse scattering transform method [1], the complex hyperbolic function method [2,3], the rank analysis method [4], the ansatz method [5,6], the Sumudu transform method [7–9], the $\exp(-\varphi(\eta))$-expansion method [10–12], the F-expansion method [13,14], the (G'/G)-expansion method [15–19], the Backlund transformation method [20], the Darboux transformation method [21], the homotopy perturbation method [22,23], the Hirota's bilinear method [24], the homogeneous balance method [25–27], the Jacobi elliptic function expansion method [28,29], new generalized (G'/G)-expansion method [30–32], the fixed pivot method [33,34] and so on.

The new generalized (G'/G) expansion method is powerful to solve NLEEs and can help to obtain many new exact solutions which we have never seen before. Within my knowledge, in this work, I will apply the new generalized (G'/G) expansion method to explore the exact solutions for the foam drainage equation.

The rest of the paper is organized as follows: In Section 2, I give the description of the new generalized (G'/G) expansion method. In Section 3, I apply this method to the foam drainage equation. In Section 4, graphical representations are given. In Section 5, discussions are given. Conclusions are given in the last section.

2. Description of the new generalized (G'/G)-expansion method

Consider the general NLEE of the type

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives. The main steps of the new generalized (G'/G) expansion method are as follows:

Step 1: The traveling wave variable ansatz

$$u(x,t) = u(\xi), \quad \xi = x \pm Vt, \tag{2}$$

where V is the speed of the traveling wave. Now using transformation (2) in Eq. (1) I obtain the following ordinary differential equation (ODE) for $u = u(\xi)$:

$$Q(u, u', u'', u''', \cdots) = 0, \tag{3}$$

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where Q is a polynomial of u and its derivatives and the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2: According to likelihood Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant might be zero, for minimalism.

Step 3: Suppose that the traveling wave solution of Eq. (3) can be expressed in the following form:

$$u(\xi) = \sum_{i=0}^{N} a_i (d+H)^i + \sum_{i=1}^{N} b_i (d+H)^{-i}, \tag{4}$$

where either a_N or b_N may be zero, but both a_N and b_N could be zero at a time, a_i $(i = 0, 1, 2, \cdots, N)$ and b_i $(i = 1, 2, \cdots, N)$ and d are arbitrary constants to be determined later and $H(\xi)$ is

$$H(\xi) = (G'/G) \tag{5}$$

where $G = G(\xi)$ satisfies the following auxiliary ordinary differential equation:

$$AGG'' - BGG' - EG^2 - C(G')^2 = 0 \tag{6}$$

where the prime stands for derivative with respect to ξ; A, B, C and E are real parameters.

Step 4: To determine the positive integer N, taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in Eq. (3).

Step 5: Substituting Eq. (4) and Eq. (6) counting Eq. (5) into Eq. (3) with the value of N obtained in Step 4, we obtain polynomials in $(d+H)^N$ $(N = 0, 1, 2, \cdots)$ and $(d+H)^{-N}$ $(N = 0, 1, 2, \cdots)$. After that, we collect each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for a_i $(i = 0, 1, 2, \cdots, N)$ and b_i $(i = 1, 2, \cdots, N), d$ and V.

Step 6: Let us consider that the value of the constants a_i $(i = 0, 1, 2, \cdots, N)$, b_i $(i = 1, 2, \cdots, N), d$ and V can be found by solving the algebraic equations obtained in Step 5. Since the general solution of Eq. (6) is well known to us, inserting the values of a_i $(i = 0, 1, 2, \cdots, N)$, b_i $(i = 1, 2, \cdots, N)$, d and V into Eq. (4), we obtain the more general type and new exact traveling wave solutions of the nonlinear partial differential Eq. (1).

Using the general solution of Eq. (6), we obtain the following solutions of Eq. (5):

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) > 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)} \tag{7}$$

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) < 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)} \tag{8}$$

When $B \neq 0$, $\psi = A - C$ and $\Omega = B^2 + 4E(A - C) = 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi} \tag{9}$$

When $B = 0$, $\psi = A - C$ and $\Delta = \psi E > 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{\Delta}}{\psi} \frac{C_1 \sinh\left(\frac{\sqrt{\Delta}}{A}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Delta}}{A}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Delta}}{A}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Delta}}{A}\xi\right)} \tag{10}$$

When $B = 0$, $\psi = A - C$ and $\Delta = \psi E < 0$,

$$H(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Delta}}{\psi} \frac{-C_1 \sin\left(\frac{\sqrt{-\Delta}}{A}\xi\right) + C_2 \cos\left(\frac{\sqrt{-\Delta}}{A}\xi\right)}{C_1 \cos\left(\frac{\sqrt{-\Delta}}{A}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{A}\xi\right)} \tag{11}$$

3. Application of the method

In this section, I will set forth the new generalized (G'/G) expansion method to erect many new and more general traveling wave solutions of the foam drainage equation. Let us consider that the foam drainage equation is,

$$u_t + \left(u^2 - \frac{\sqrt{u}}{2} u_x\right)_x = 0. \tag{12}$$

where x and t are scaled position and time coordinates, respectively and u is the cross section of a channel formed where three films meet, usually indicated as Plateau border. In this paper, we show the usefulness and convenience of the method by obtaining the exact solution of Eq. (12). Foam is central to a number of everyday activities, both natural and industrial. Recent research in foams has centered on three topics which are often treated separately, but are, in fact, interdependent: drainage, coarsening and rheology. We concentrate on a quantitative description of the coupling of drainage. The flow of liquid through Plateau borders (the liquid-filled channels) and intersections of four channels between the bubbles, driven by gravity and capillarity, is called foam drainage. Foams' drainage plays a very important role in foam stability. In fact, when foam dries, its structure becomes fragile (see, Ref. [35]). In spite of many applications and numerous scientific investigations of properties and mechanics of foams, dynamics of foam drainage have only recently been examined in detail. (Ref. [36]) used a semi-analytical method, that is the Adomian decomposition and the tanh method to handle the foam drainage Eq. (12). Also, Eq. (12) was studied by another author using different methods, such as homotopy analysis method (Ref. [37]) and the variational approach (Ref. [38]). Foams are of great importance in many technological processes and applications, and their properties are the subject of intensive studies from both practical and scientific points of view. This is why foam has been of great interest for academic research. Because of the everyday occurrence of foams, they are very well known to scientists as well as to common people (Ref. [39,40]). Foams are common in foods and personal care products such as lotions and creams and foams often occur during cleaning of clothes and scrubbing (see, Ref. [41]). They have important applications in food and chemical industries, mineral processing, fire fighting and structural material sciences (see, for example, Ref. [42]). Everyday experiences put us in direct contact with foams. Shampooing hair, washing dishes, eating chocolate bars and chocolate mousse desserts are only a few examples. History connects foams with a number of famous scientists and foam continues to excite imaginations (Ref. [43]).

I utilize the traveling wave variable $u(\xi) = u(x, t)$, $\xi = k(x + Vt)$, Eq. (12) is carried to an ODE

$$Vku' + k\left(u^2 - \frac{k}{2}\sqrt{u}u'\right)' = 0. \tag{13}$$

Integrating (13) with respect to ξ and considering the zero constants for integration we obtain

$$Vku + k\left(u^2 - \frac{k}{2}\sqrt{u}u'\right) = 0 \tag{14}$$

then I use the transformation

$$u(\xi) = v^2(\xi), \tag{15}$$

that will convert Eq. (14) to

$$kVv^2 + kv^4 - k^2 v^2 v' = 0, \tag{16}$$

or equivalently

$$V + v^2 - kv' = 0, \tag{17}$$

Taking the homogeneous balance between highest order nonlinear term v^2 and linear term of the highest order v' in Eq. (17), we obtain $N = 1$. Therefore, the solution of Eq. (17) is of the form:

$$v(\eta) = \alpha_0 + \alpha_1(d+M) + \beta_1(d+M)^{-1}, \tag{18}$$

where $\alpha_0,\ \alpha_1,\ \beta_1$ and d are constants to be determined.

Substituting Eq. (18) together with Eqs. (5) and (6) into Eq. (17), the left-hand side is converted into polynomials in $(d+H)^N$ $(N = 0, 1, 2, \ldots\ldots)$ and $(d+H)^{-N}$ $(N = 1, 2, \cdots)$. I collect each coefficient of these resulted polynomials to zero yields a set of simultaneous algebraic equations (for simplicity, the equations are not presented) for $\alpha_0,\ \alpha_1,\ \beta_1,\ d,\ P$ and V. Solving these algebraic equations with the help of computer algebra, I obtain the following:

$$\textbf{Set 1}: \alpha_0 = -\frac{k}{2A}(B+2d\psi),\ \alpha_1 = 0,\ \beta_1 = \frac{k}{A}(d^2\psi + Bd - E),$$
$$d = d,\ V = -\frac{k^2}{4A^2}(4E\psi + B^2), \tag{19}$$

where $\psi = A - C,\ d,\ A,\ B,\ C,\ E$ are free parameters.

$$\textbf{Set 2}: \alpha_0 = \frac{k}{2A}(B+2d\psi), V = -\frac{k^2}{4A^2}(4E\psi + B^2), d = d, \beta_1 = 0,$$
$$\alpha_1 = -\frac{k\psi}{A}. \tag{20}$$

where $\psi = A - C,\ d,\ A,\ B,\ C,\ E$ are free parameters.

$$\textbf{Set 3}: \alpha_0 = 0, \alpha_1 = -\frac{k\psi}{A}, V = -\frac{k^2}{A^2}(4E\psi + B^2), d = \frac{-B}{2\psi},$$
$$\beta_1 = -\frac{k}{4A\psi}(4E\psi + B^2), \tag{21}$$

where $\psi = A - C,\ d,\ A,\ B,\ C,\ E$ are free parameters.

For set 1, substituting Eq. (19) into Eq. (18), along with Eq. (7) and simplifying, yields the following traveling wave solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{1_1}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-1}\right\}.$$

and

$$u_{1_1}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2\left(d^2\psi + Bd - E\right) \times \left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-1}\right\}\right]^2.$$

$$v_{1_2}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2\left(d^2\psi + Bd - E\right) \times \left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-1}\right\}.$$

and

$$u_{1_2}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi}\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right)^{-1}\right\}\right]^2.$$

Substituting Eq. (19) into Eq. (18), along with Eq. (8) and simplifying, our exact solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{1_3}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi}\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-1}\right\}.$$

and

$$u_{1_3}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi}\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-1}\right\}\right]^2.$$

$$v_{1_4}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi}\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-1}\right\}.$$

and

$$u_{1_4}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi}\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right)^{-1}\right\}\right]^2.$$

Substituting Eq. (19) into Eq. (18), together with Eq. (9) and simplifying, our obtained solution becomes:

$$v_{1_5}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi}\right)^{-1}\right\}.$$

and

$$u_{1_5}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{B}{2\psi} + \frac{C_2}{C_1 + C_2\xi}\right)^{-1}\right\}\right]^2.$$

Substituting Eq. (19) into Eq. (18), along with Eq. (10) and simplifying, we obtain the following traveling wave solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{1_6}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1}\right\}.$$

and

$$u_{1_6}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1}\right\}\right]^2.$$

$$v_{1_7}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1}\right\}.$$

and

$$u_{1_7}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1}\right\}\right]^2$$

Substituting Eq. (19) into Eq. (18), together with Eq. (11) and simplifying, our obtained exact solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{1_8}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1}\right\},$$

and

$$u_{1_8}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d + \frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1}\right\}\right]^2$$

$$v_{1_9}(\xi) = -\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d - \frac{\sqrt{-\Delta}}{\psi}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1}\right\},$$

and

$$u_{1_9}(\xi) = \left[-\frac{k}{2A}\left\{(B+2d\psi) - 2(d^2\psi + Bd - E) \times \left(d - \frac{\sqrt{-\Delta}}{\psi}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1}\right\}\right]^2$$

where $\xi = x - \left\{-\frac{k^2}{4A^2}(4E\psi + B^2\right\}t$.

Again for set 2, substituting Eq. (20) into Eq. (18), along with Eq. (7) and simplifying, our traveling wave solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{2_1}(\xi) = -\frac{k}{2A}\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right),$$

and

$$u_{2_1}(\xi) = \left[-\frac{k}{2A}\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right]^2,$$

$$v_{2_2}(\xi) = -\frac{k}{2A}\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right),$$

and

$$u_{2_2}(\xi) = \left[-\frac{k}{2A}\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right]^2,$$

Substituting Eq. (20) into Eq. (18), along with Eq. (8) and simplifying yields exact solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{2_3}(\xi) = -\frac{ik}{2A}\sqrt{\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right),$$

and

$$u_{2_3}(\xi) = \left[-\frac{ik}{2A}\sqrt{\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right]^2,$$

$$v_{2_4}(\xi) = \frac{ik}{2A}\sqrt{\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right),$$

and

$$u_{2_4}(\xi) = \left[\frac{ik}{2A}\sqrt{\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right]^2,$$

Substituting Eq. (20) into Eq. (18), along with Eq. (9) and simplifying, our obtained solution becomes:

$$v_{2_5}(\xi) = -\frac{k\psi}{A}\left(\frac{C_2}{C_1 + C_2\xi}\right),$$

and

$$u_{2_5}(\xi) = \left[-\frac{k\psi}{A}\left(\frac{C_2}{C_1 + C_2\xi}\right)\right]^2,$$

Substituting Eq. (20) into Eq. (18), together with Eq. (10) and simplifying, yields the following traveling wave solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$, respectively:

$$v_{2_6}(\xi) = \frac{k}{2A}\left(B - 2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right).$$

and

$$u_{2_6}(\xi) = \left[\frac{k}{2A}\left(B - 2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)\right]^2$$

$$v_{2_7}(\xi) = \frac{k}{2A}\left(B - 2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right).$$

and

$$u_{2_7}(\xi) = \left[\frac{k}{2A}\left(B - 2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)\right]^2$$

Substituting Eq. (20) into Eq. (18), along with Eq. (11) and simplifying, our exact solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$, respectively:

$$v_{2_8}(\xi) = \frac{k}{2A}\left(B + 2i\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)$$

and

$$u_{2_8}(\xi) = \left[\frac{k}{2A}\left(B + 2i\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)\right]^2$$

$$v_{2_9}(\xi) = \frac{k}{2A}\left(B - 2i\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right),$$

and

$$u_{2_9}(\xi) = \left[\frac{k}{2A}\left(B - 2i\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)\right]^2.$$

where $\xi = x - \left\{-\frac{k^2}{4A^2}(4E\psi + B^2\right\}t$.

Similarly, for set 3, substituting Eq. (21) into Eq. (18), together with Eq. (7) and simplifying, yields the following traveling wave solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$, respectively:

$$v_{3_1}(\xi) = -\frac{k}{2A}\left\{\sqrt{\Omega}\times\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + \frac{1}{\sqrt{\Omega}}(4E\psi + B^2)\times\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right\}.$$

and

$$u_{3_1}(\xi) = \left[-\frac{k}{2A}\left\{\sqrt{\Omega}\times\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + \frac{1}{\sqrt{\Omega}}(4E\psi + B^2)\times\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right\}\right]^2$$

$$v_{3_2}(\xi) = -\frac{k}{2A}\left\{\sqrt{\Omega}\times\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + \frac{1}{\sqrt{\Omega}}(4E\psi + B^2)\times\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right\}.$$

and

$$u_{3_2}(\xi) = \left[-\frac{k}{2A}\left\{\sqrt{\Omega}\times\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + \frac{1}{\sqrt{\Omega}}(4E\psi + B^2)\times\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right\}\right]^2$$

Substituting Eq. (21) into Eq. (18), along with Eq. (8) and simplifying, we obtain the following solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$v_{3_3}(\xi) = -\frac{k}{2A}\left\{i\sqrt{\Omega}\times\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + \frac{1}{i\sqrt{\Omega}}(4E\psi + B^2)\times\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right\}.$$

and

$$u_{3_2}(\xi) = \left[-\frac{k}{2A}\left\{\sqrt{\Omega}\times\tanh\left(\frac{\sqrt{\Omega}}{2A}\xi\right) + \frac{1}{\sqrt{\Omega}}(4E\psi + B^2)\times\coth\left(\frac{\sqrt{\Omega}}{2A}\xi\right)\right\}\right]^2$$

$$v_{3_4}(\xi) = \frac{k}{2A}\left\{i\sqrt{\Omega}\times\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + \frac{1}{i\sqrt{\Omega}}(4E\psi + B^2)\times\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right\}.$$

and

$$u_{3_4}(\xi) = \left[\frac{k}{2A}\left\{i\sqrt{\Omega}\times\tan\left(\frac{\sqrt{-\Omega}}{2A}\xi\right) + \frac{1}{i\sqrt{\Omega}}(4E\psi + B^2)\times\cot\left(\frac{\sqrt{-\Omega}}{2A}\xi\right)\right\}\right]^2$$

Substituting Eq. (21) into Eq. (18), along with Eq. (9) and simplifying, our obtained solution becomes:

$$v_{3_5}(\eta) = -\frac{k}{A}\left\{\psi\times\left(\frac{C_2}{C_1 + C_2\xi}\right) + \frac{1}{4\psi}(4E\psi + B^2)\times\left(\frac{C_2}{C_1 + C_2\xi}\right)^{-1}\right\}$$

and

$$u_{3_5}(\eta)=\left[-\frac{k}{A}\left\{\psi\times\left(\frac{C_2}{C_1+C_2\xi}\right)+\frac{1}{4\psi}(4E\psi+B^2)\times\left(\frac{C_2}{C_1+C_2\xi}\right)^{-1}\right\}\right]^2$$

Substituting Eq. (21) into Eq. (18), along with Eq. (10) and simplifying, yields following exact traveling wave solutions, if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$ respectively:

$$v_{3_6}(\xi)=-\frac{k\psi}{A}\times\left(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)-\frac{k}{4A\psi}(4E\psi+B^2)\times\left(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1}.$$

and

$$u_{3_6}(\xi)=\left[-\frac{k\psi}{A}\times\left(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)-\frac{k}{4A\psi}(4E\psi+B^2)\times\left(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\coth\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1}\right]^2$$

$$v_{3_7}(\xi)=-\frac{k\psi}{A}\times\left(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\eta\right)\right)-\frac{k}{4A\psi}(4E\psi+B^2)\times\left(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1}.$$

and

$$u_{3_7}(\xi)=\left[-\frac{k\psi}{A}\times\left(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\eta\right)\right)-\frac{k}{4A\psi}(4E\psi+B^2)\times\left(\frac{-B}{2\psi}+\frac{\sqrt{\Delta}}{\psi}\tanh\left(\frac{\sqrt{\Delta}}{A}\xi\right)\right)^{-1}\right]^2$$

Substituting Eq. (21) into Eq. (18), along with Eq. (11) and simplifying, our obtained exact solutions become, if $C_1=0$ but $C_2\neq 0$; $C_2=0$ but $C_1\neq 0$ respectively:

$$v_{3_8}(\xi)=-\frac{k\psi}{A}\times\left(\frac{-B}{2\psi}+\frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi)\right)-\frac{k}{4A\psi}(4E\psi+B^2)\times\left(\frac{-B}{2\psi}+\frac{\sqrt{-\Delta}}{\psi}\cot\left(\frac{\sqrt{-\Delta}}{A}\xi\right)\right)^{-1}$$

and

$$u_{3_8}(\xi)=\left[-\frac{k\psi}{A}\times\left(\frac{-B}{2\psi}+\frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi)\right)-\frac{k}{4A\psi}(4E\psi+B^2)\times\left(\frac{-B}{2\psi}+\frac{\sqrt{-\Delta}}{\psi}\cot(\frac{\sqrt{-\Delta}}{A}\xi)\right)^{-1}\right]^2$$

$$v_{3_9}(\xi)=-\frac{k\psi}{A}\times\left(\frac{-B}{2\psi}-\frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi)\right)-\frac{k}{4A\psi}(4E\psi+B^2)\times\left(\frac{-B}{2\psi}-\frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi)\right)^{-1}$$

and

$$u_{3_9}(\xi)=\left[-\frac{k\psi}{A}\times\left(\frac{-B}{2\psi}-\frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi)\right)-\frac{k}{4A\psi}(4E\psi+B^2)\times\left(\frac{-B}{2\psi}-\frac{\sqrt{-\Delta}}{\psi}\tan(\frac{\sqrt{-\Delta}}{A}\xi)\right)^{-1}\right]^2$$

where $\xi=x-\left\{-\frac{k^2}{A^2}(4E\psi+B^2)\right\}t.$

4. Graphical presentation

Graph is a powerful tool for communication and describes lucidly the solutions of the problems. Therefore, some graphs of the solutions are given below. The graphs readily have shown the solitary wave form of the solutions (see Figs. 1–10).

5. Discussions

The advantages and validity of the method over the basic (G'/G)-expansion method have been discussed in the following.

5.1. Advantages

The significant advantage of the new generalized (G'/G)-expansion method over the basic (G'/G)-expansion method is that the method provides more general and a huge amount of new exact traveling wave solutions with numerous free parameters. The

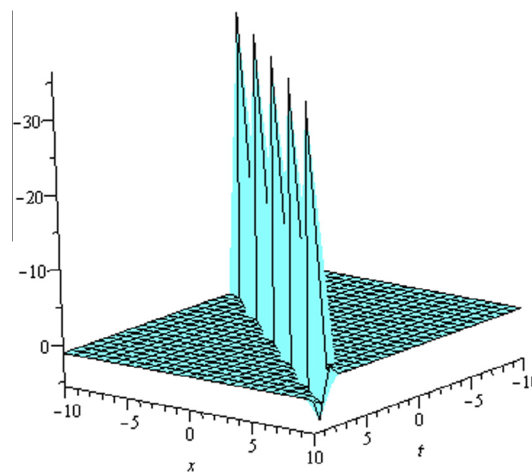

Fig. 1a. Soliton corresponding to solution $v_{1_1}(\xi)$ for $k=2,\ d=1,\ A=2,\ B=1,\ C=1,\ E=1$ and $-10\leqslant x,t\leqslant 10$.

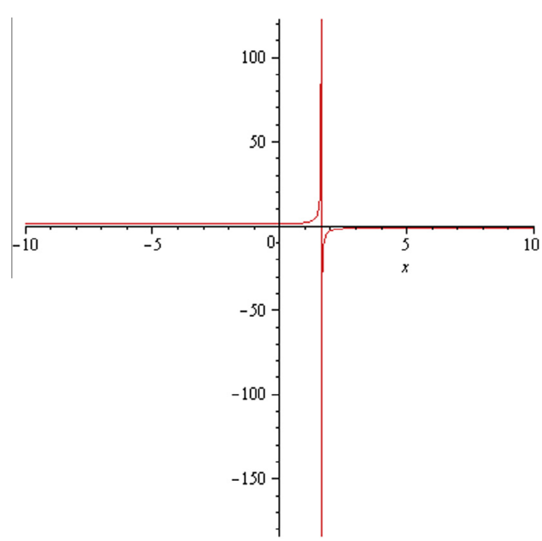

Fig. 1b. Soliton corresponding to solution $v_{1_1}(\xi)$ for $t=2,\ k=2,\ d=1,\ A=2,\ B=1,\ C=1$ and $E=1$.

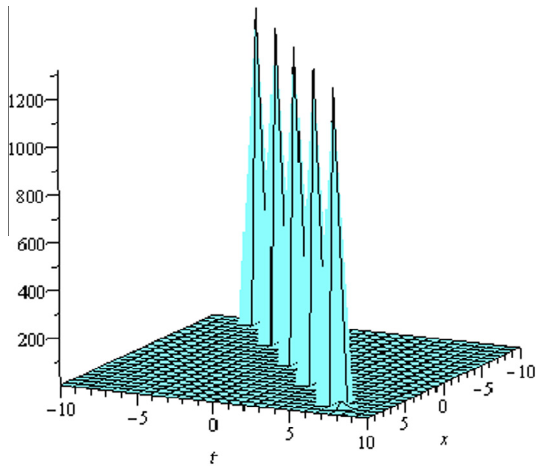

Fig. 2a. Soliton corresponding to solution $u_{1_1}(\xi)$ for $k=2,\ d=1,\ A=2,\ B=1,\ C=1,\ E=1$ and $-10 \leqslant x,t \leqslant 10$.

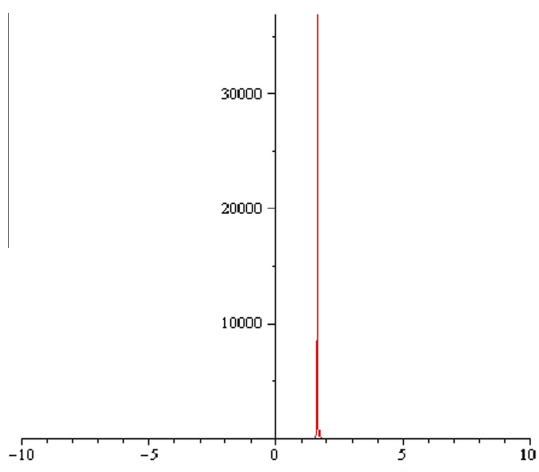

Fig. 2b. Soliton corresponding to solution $u_{1_1}(\xi)$ for $t=2,\ k=2,\ d=1,\ A=2,\ B=1,\ C=1$ and $E=1$.

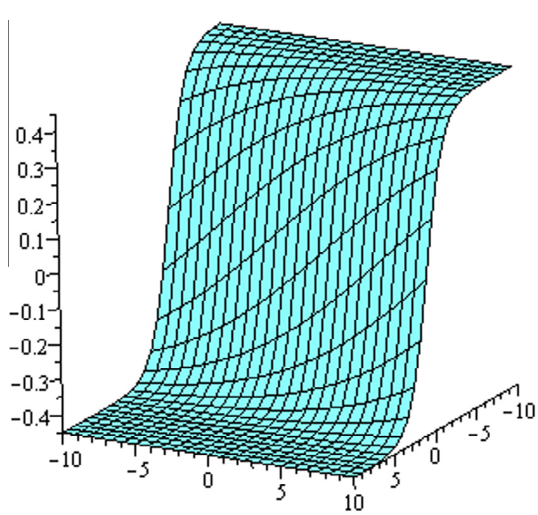

Fig. 3a. Soliton corresponding to solution $v_{1_2}(\xi)$ for $k=1,\ d=1,\ A=4,\ B=1,\ C=1,\ E=1$ and $-10 \leqslant x,t \leqslant 10$.

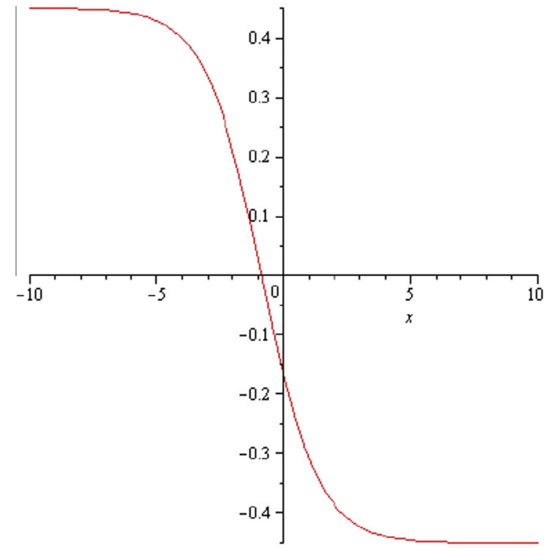

Fig. 3b. Soliton corresponding to solution $v_{1_2}(\xi)$ for $t=2,\ k=1,\ d=1,\ A=4,\ B=1,\ C=1$ and $E=1$.

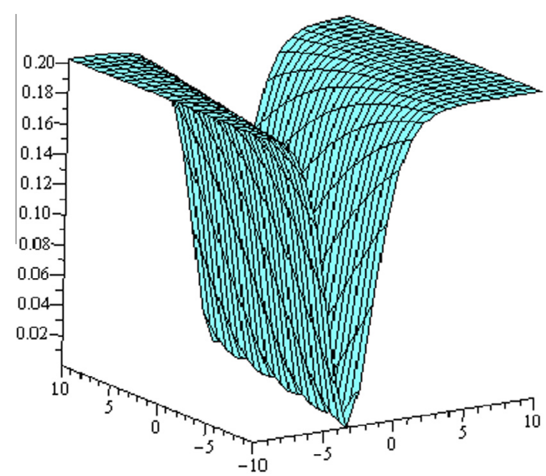

Fig. 4a. Soliton corresponding to solution $u_{1_2}(\xi)$ for $k=1,\ d=1,\ A=4,\ B=1,\ C=1,\ E=1$ and $-10 \leqslant x,t \leqslant 10$.

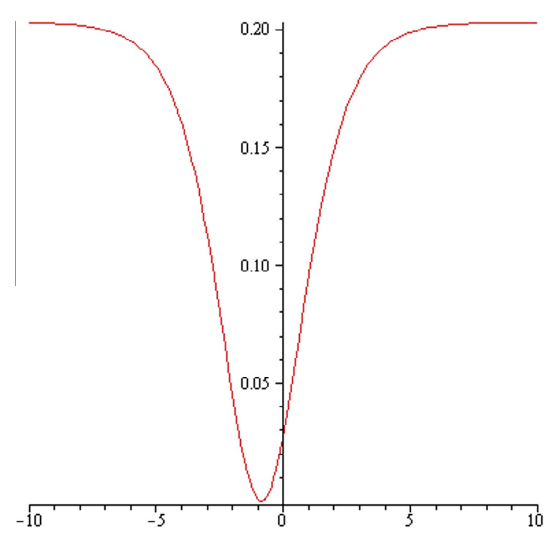

Fig. 4b. Soliton corresponding to solution $u_{1_2}(\xi)$ for $t=2,\ k=1,\ d=1,\ A=4,\ B=1,\ C=1$ and $E=1$.

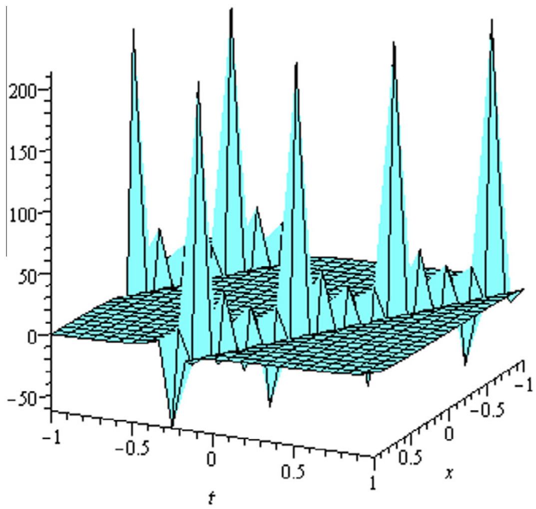

Fig. 5a. Soliton corresponding to solution $v_{1_3}(\xi)$ for $k = 2,\ d = 1,\ A = 2,\ B = 1,\ C = 4,\ E = 1$ and $-1 \leqslant x, t \leqslant 1$.

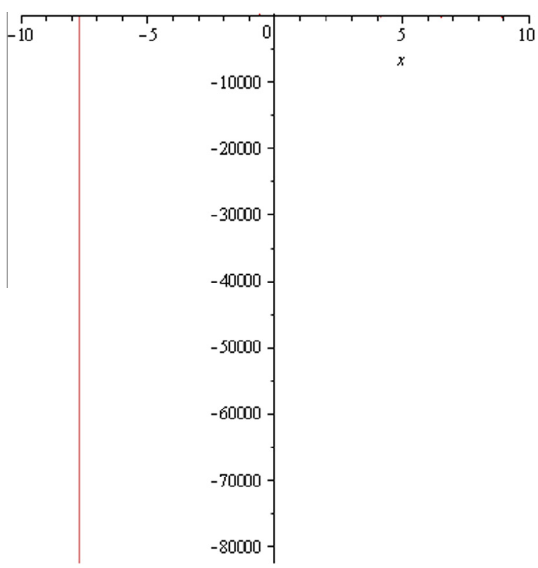

Fig. 5b. Soliton corresponding to solution $v_{1_3}(\xi)$ for $t = 2,\ k = 2,\ d = 1,\ A = 2,\ B = 1,\ C = 4$ and $E = 1$.

exact solutions have its huge meaning to expose the internal instrument of the complex physical phenomena. Apart from the physical application, the close-form solutions of NLEEs assist the numerical solvers to compare the correctness of their results and help them in the stability analysis.

5.2. Validity

In Ref. [44] Bekir and Uygun used the linear ordinary differential equation as auxiliary equation and traveling wave solutions presented in the form $u(\xi) = \sum_{i=0}^{m} a_i (G'/G)^i$, where $a_m \neq 0$. It is remarkable to point out that some of my solutions coincided with previous published results, if parameters take particular values which authenticate my solutions. Moreover, in Ref. [44] Bekir and Uygun investigated the well-established the foam drainage equation to obtain exact solutions via the basic (G'/G)-expansion method and achieved only three solutions (A.1)–(A.3) (see Appendix). Moreover, abundant traveling wave solutions of the

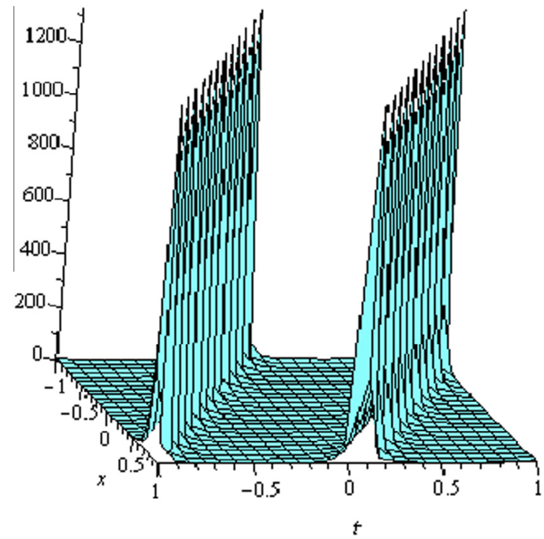

Fig. 6a. Soliton corresponding to solution $u_{2_9}(\xi)$ for $k = 1,\ A = 1,\ B = 0,\ C = 2,\ E = 2$ and. $-1 \leqslant x, t \leqslant 1$.

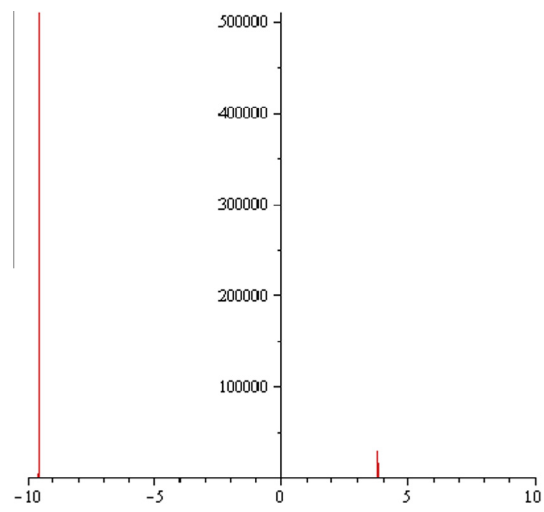

Fig. 6b. Soliton corresponding to solution $u_{2_9}(\xi)$ for $t = 2,\ k = 1,\ A = 1,\ B = 0,\ C = 2$ and $E = 2$.

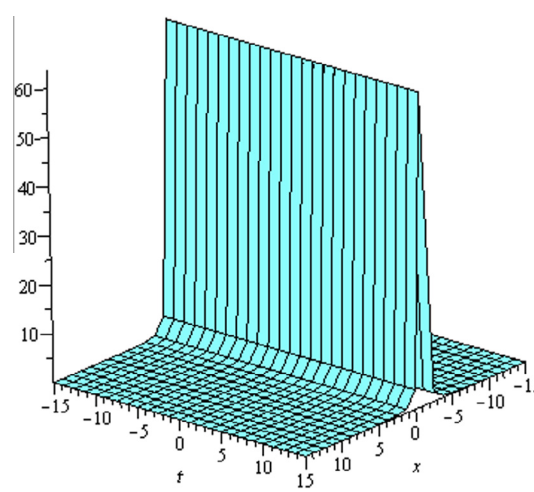

Fig. 7a. Soliton corresponding to solution $u_{2_5}(\xi)$ for $k = 2,\ C_1 = 1,\ C_2 = 2,\ A = 1,\ B = 2,\ C = 2,\ E = 1$ and $-15 \leqslant x, t \leqslant 15$.

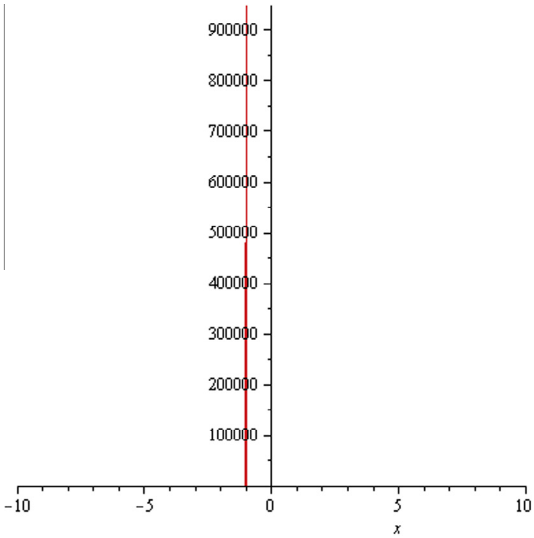

Fig. 7b. Soliton corresponding to solution $u_{2_5}(\xi)$ for $t=2,\ k=2,\ A=1,\ B=2,\ C=2$ and $E=1$.

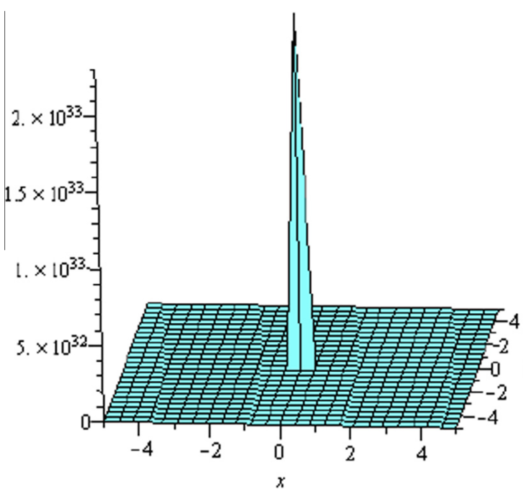

Fig. 8a. Soliton corresponding to solution $u_{2_1}(\xi)$ for $t=2,\ k=2,\ A=4,\ B=1,\ C=1,\ E=1$ and $-5\leqslant x,t\leqslant 5$.

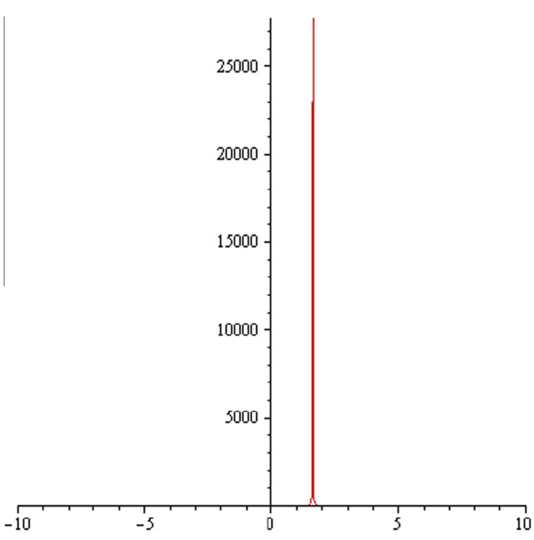

Fig. 8b. Soliton corresponding to solution $u_{2_1}(\xi)$ for $t=2,\ k=2,\ A=4,\ B=1,\ C=1$ and $E=1$.

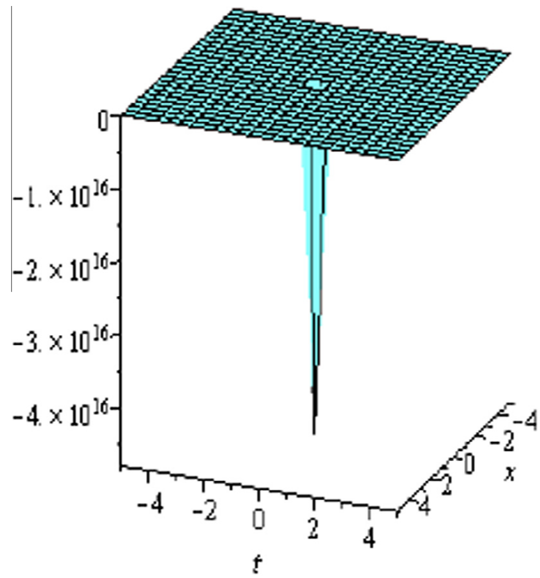

Fig. 9a. Soliton corresponding to solution $v_{2_1}(\xi)$ for $t=2,\ k=2,\ A=4,\ B=1,\ C=1,\ E=1$ and $-5\leqslant x,t\leqslant 5$.

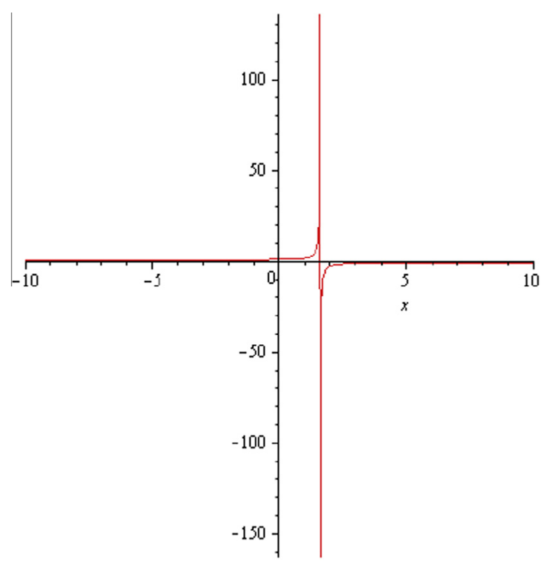

Fig. 9b. Soliton corresponding to solution $v_{2_1}(\xi)$ for $t=2,\ k=2,\ A=4,\ B=1,\ C=1$ and $E=1$.

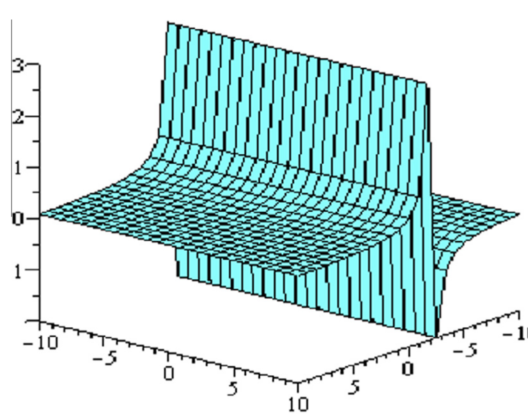

Fig. 10a. Soliton corresponding to solution $v_{3_5}(\xi)$ for $k=1,\ C_1=1,\ C_2=2,\ A=1,\ B=2,\ C=2,\ E=1$ and $-10\leqslant x,t\leqslant 10$.

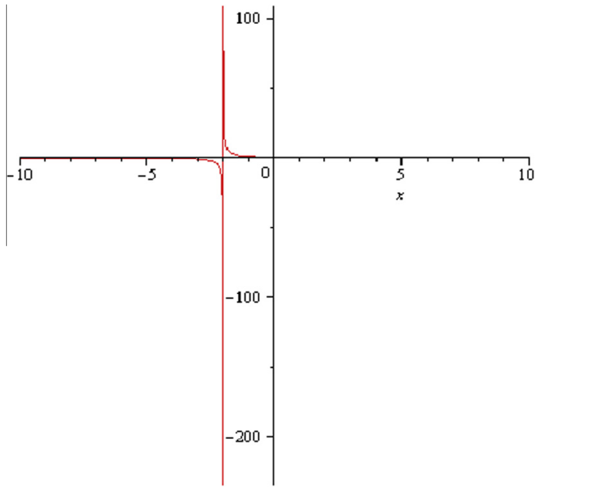

Fig. 10b. Soliton corresponding to solution $v_{3_5}(\xi)$ for $t=2,\ k=1,\ C_1=1,\ C_2=2,\ A=1,\ B=2,\ C=2,\ E=1$.

well-known foam drainage equation are constructed by using the new generalized (G'/G)-expansion method.

6. Conclusion

The new generalized (G'/G)-expansion method offered in this paper has been fruitfully implemented to put up many new and more general exact solutions of the foam drainage equation. The method offers solutions with free parameters that might be important to enlighten some intricate physical phenomena. Comparing the currently proposed method with other methods, such as (G'/G)-expansion method, the Exp-function method and the MSE method, I might conclude that the exact solutions to Eq. (12) can be investigated using these methods with the help of the symbolic computation software such as Matlab, Mathematica and Maple to facilitate the complicated algebraic computations. This study shows that the method is quite well-organized and practically well suited to be used in finding exact solutions of NLEEs. Also, I observe that the method is straightforward and can be applied to many other NLEEs.

Appendix: Bekir and Uygun solutions [44]

Bekir and Uygun [44] established exact solutions of the well-known foam drainage equation by using the basic (G'/G)-expansion method which is as follows:

When $\lambda^2-4\mu>0$,

$$u_1=\left[-k\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)\right]^2, \tag{A.1}$$

where $\xi=k\left[x+\frac{k^2}{4}(4\mu-\lambda^2)\right]t$ and C_1, C_2 are arbitrary constants.

When $\lambda^2-4\mu<0$,

$$u_2=\left[-k\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right)\right]^2, \tag{A.2}$$

where $\xi=k\left[x+\frac{k^2}{4}(4\mu-\lambda^2)\right]t$ and C_1, C_2 are arbitrary constants.

When $\lambda^2-4\mu=0$,

$$u_3=\frac{k^2C_2^2}{(C_1+C_2kx)^2}+a_0, \tag{A.3}$$

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